AN ANALYSIS OF METHODS FOR WAVEFRONT RECONSTRUCTION FROM GRADIENT MEASUREMENTS IN ADAPTIVE OPTICS

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Abstract

The use of adaptive optics (AO) in ground-based astronomy is becoming increasingly mainstream. While classical methods, such as deconvolution, remove the blur in an image only after it has been collected, AO systems seek to remove phase error in incoming wavefronts prior to image formation, resulting in higher resolution images. If the phase error is known, it can be removed via the creation of a counter wavefront using, e.g., a deformable mirror. In the AO systems used on ground-based telescopes, an estimate of the phase error is typically obtained by solving an inverse problem involving measurements of the wavefront gradient. The standard approach for obtaining phase estimates from measurements of its gradient is least squares. However, a more robust solution can be obtain if a minimum variance, or penalized least squares, approach is taken instead. In this paper, we will perform a theoretical analysis of these approaches in a continuous, i.e. function space, setting.

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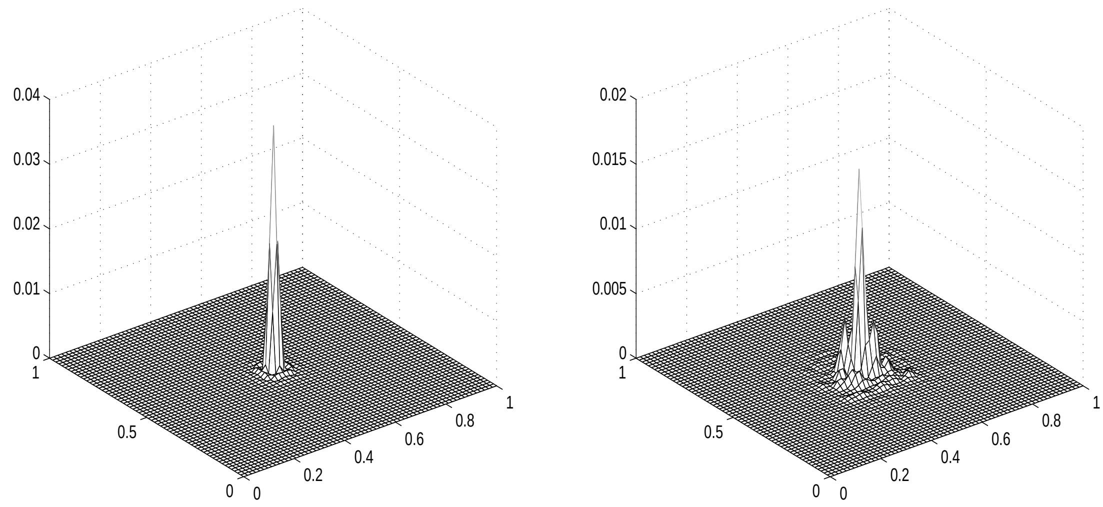

Figure 1: Diffraction limited PSF (on the left), and PSF resulting from physically realistic phase error (on the right).

1 Introduction

The classical approach for removing blur from an image d collected by a ground-based telescope is to solve a deconvolution problem of the form

$$d(x,y) = \int_{\mathbb{R}^2} k(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta \tag{1}$$

for f given the *point spread function* (PSF) k. This problem has seen, and continues to see, a great deal of attention in the mathematical community; a large body of theoretical analysis and computational methodology has been developed for its solution (see, e.g., [11, 12] and the reference therein). What makes solving (1) difficult, but also mathematically interesting, is that it is an ill-posed problem. However for the astronomer, mathematical interest is a secondary consideration. Thus it should be no surprise that astronomers have sought image enhancement techniques that involve the solution of well-posed problems instead. The recent and resounding success of adaptive optics (AO) [2] is proof that astronomers have been successful in this endeavor.

In order to properly motivate AO methodology, we introduce the spatially invariant, Fourier optics PSF model [10]

$$k[\phi](x,y) = \left|\mathcal{F}^{-1}\left\{\mathcal{M}(x,y)e^{i\phi(x,y)}\right\}\right|^2. \tag{2}$$

Here $\mathcal{M}(x,y)$ is the telescope's pupil indicator function, i.e. is 1 inside the pupil and 0 otherwise; it characterizes the diffractive blur in the imaging system. The function $\phi(x,y)$, which characterizes refractive blur, denotes the phase error, or simply the phase, and is determined by the deviation from planarity of the incoming wavefronts of light at the point (x,y). In Figure 1, we plot (2) in the *diffraction limited* case, i.e. when $\phi = 0$, and with a nonzero ϕ generated by a physically realistic model. Note the negative effect of phase error on the right-hand side PSF.

The job of the AO system is to remove the phase error from incoming wavefronts by introducing a counter wavefront ϕ_{DM} via, e.g., a deformable mirror [2]. Assuming the PSF has the form (2), the phase corrected PSF will have the form

$$k[\phi+\phi_{\mathrm{DM}}](x,y) = \left|\mathcal{F}^{-1}\left\{\mathcal{M}(x,y)e^{i(\phi+\phi_{\mathrm{DM}})(x,y)}\right\}\right|^2. \tag{3}$$

Ideally, the deformable mirror created counter wavefront satisfies $\phi_{\mathrm{DM}} = -\phi$, so that the resulting PSF has the *diffraction limited* form

$$k[0](x,y) = \left|\mathcal{F}^{-1}\left\{\mathcal{M}(x,y)\right\}\right|^2, \tag{4}$$

in which case the *diffraction limited image*

$$d_{\mathrm{DL}}(x,y) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^2} k[0](x-\xi, y-\eta) f(\xi,\eta)\, d\xi d\eta, \tag{5}$$

– the astronomers gold standard – is what is seen by the telescope. In practice, however, an accurate approximation of $-\phi$ suffices for near diffraction limited imaging.

In ground-based astronomy, phase estimates are typically obtained from measurements of the wavefront gradient, in which case, the the following inverse problem must be solved for ϕ:

$$g = \mathcal{M}\nabla\phi + n, \quad \text{on} \quad \Omega. \tag{6}$$

Here Ω is the computational domain, $\mathcal{M}$ is the pupil indicator function mentioned above, g denotes the measured gradient, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$ is the gradient operator, and n denotes measurement error.

Before continuing, we note that there exist techniques for estimating phase that do not use measurements of the gradient. Important examples include curvature sensing, self-referencing interferometry, normalized cross-correlation, and phase diversity (cf. [6, 9] for details). However, in this paper, we focus on problem (6).

The classical approaches for solving (6) (cf. [3, 7, 8]) correspond, in the continuous setting, to the problem of minimizing the least squares functional

$$J_0(\phi) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega (\mathcal{M}\nabla\phi - g)^2 dx\, dy. \tag{7}$$

It has been observed, however, that more accurate and stable phase estimates can be obtained if the minimum variance estimate is computed instead. Minimum variance estimation requires a prior probability density function for the unknown phase ϕ. In astronomical adaptive optics, the standard choice of prior is the Gaussian probability density

$$p_\phi(\phi) = \exp\left\{\frac{1}{2}\int_\Omega \left(C_\phi^{-1/2}\phi\right)^2 dx\, dy\right\}, \tag{8}$$

where the covariance operator C_ϕ is given by the Kolmogorov atmospheric turbulence model

$$C_\phi = \mathcal{F}^*\Lambda(\omega)\mathcal{F}, \tag{9}$$

where $\mathcal{F}$ and $\mathcal{F}^*$ are the two-dimensional Fourier and inverse Fourier transforms respectively, and

$$\Lambda(\omega) = \frac{c_0}{(|\omega|^2 + 1/L_0^2)^{11/6}}. \tag{10}$$

Here L_0 is the turbulence outer-scale, which prevents an unphysically infinite amount of energy at the origin, and c_0 is the phase screen strength (c.f. [10]). However in order to facilitate faster computations, Ellerbroek [4] proposed approximating C_ϕ in (9), (10) by

$$C_\phi^{-1} = (1/c_0)\Delta^2. \tag{11}$$

Here the Kolmogorov power spectral density (10) is approximated as follows: set $L_0 = \infty$ and note that

$$|\omega|^{-11/3} \approx |\omega|^{-4}.$$

Approximation (11) is then obtained by noting that the biharmonic, or squared Laplacian, operator Δ^2 has spectrum $|k|^4$. When the prior is given by (8), (11), and n in (6) is assumed to be Gaussian white noise with zero mean and variance σ^2, the minimum variance estimate is the minimizer of the functional

$$J_\sigma(\phi) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega (\mathcal{M}\nabla\phi - g)^2 dx\, dy + \frac{\sigma^2}{c_0}\int_\Omega (\Delta\phi)^2 dxdy. \tag{12}$$

A third approach for obtaining phase reconstructions is given in [1] and is derived from the Euler-Lagrange equations for (12). It is given in the continuous setting as follows: first compute the minimizer ϕ_{MNLS} of (7) with minimum $L^2(\Omega)$ norm; then compute the minimizer of the functional

$$\int_\Omega (\phi - \phi_{\text{MNLS}})^2\, dx\, dy + (\sigma^2/c_0)\int_\Omega (\nabla\phi)^2\, dx\, dy, \tag{13}$$

This approach was shown to be effective in practice in [1]. Due to the form of (13), we call this approach gradient denoised least squares (GDLS).

Our goal in this paper is to perform a theoretical analysis of the problems of minimizing the functionals (7), (12), and (13). In particular, we will show that each is a well-posed problem, and hence, is stable with respect to both modelling and stochastic errors. Well-posedness results are not only academic since in practice, the mathematical and statistical models used are only approximate. We will also prove that the minimizers of (12) and (13) converge as $\sigma^2 \to 0$; note that $\sigma^2 = 0$ corresponds to complete confidence in model (6). Finally, we will argue that the minimizers of (12) and (13) can be expected to more closely resemble true atmospheric phase profiles than do the minimizers of (7).

2 Theoretical Analysis

We begin by stating our assumptions and making necessary definitions. We assume that Ω is open and bounded with smooth boundary. In our analysis, we will make reference to the following Sobolev spaces

$$H_0^1(\Omega) = \left\{\phi \in H^1(\Omega) \mid \phi = 0 \quad \text{on} \quad \partial\Omega)\right\},$$

and

$$H_0^2(\Omega) \stackrel{\text{def}}{=} \left\{\phi \in H^2(\Omega) \;\middle|\; \phi = \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0 \text{ on } \partial\Omega\right\},$$

where

$$H^1(\Omega) = \left\{ \phi \in L^2(\Omega) \;\middle|\; \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y} \in L^2(\Omega) \right\},$$

and

$$H^2(\Omega) \stackrel{\text{def}}{=} \left\{ \phi \in L^2(\Omega) \;\middle|\; \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial^2\phi}{\partial x^2}, \frac{\partial^2\phi}{\partial x\partial y}, \frac{\partial^2\phi}{\partial y^2} \in L^2(\Omega) \right\}. \tag{14}$$

The derivatives of ϕ above are meant in the weak sense [5].

Atmospheric turbulence statistics suggest [10] that assuming that the true phase $\phi_{\text{true}} \in H^2(\Omega)$ is accurate. Noting, furthermore, that any constant offset in ϕ_{true} will have no effect on image quality, we can make the additional assumption that the offset is zero and hence that $\phi_{\text{true}} = 0$ on $\partial\Omega$. It is also the case in practice that the linear off-set, or tip-tilt, in ϕ_{true} is estimated and corrected in a separate process [2], allowing us to assume that the tip-tilt is zero, and hence, that $\partial\phi_{\text{true}}/\partial dx = \partial\phi_{\text{true}}/\partial dy = 0$ on $\partial\Omega$. Taking all of these observation together yields $\phi_{\text{true}} \in H_0^2(\Omega)$, which motives our desire to obtain phase estimates contained in $H_0^2(\Omega)$.

Finally, we note that a problem is well-posed provided it admits a unique solution that depends continuously on the data given in the problem. In the context of wavefront reconstruction, results of well-posedness for a particular approach imply that even if wavefront gradient measurements contain both modelling and stochastic errors, the corresponding wavefront estimate will be stable with respect to these errors.

Our first theoretical result deals with the least squares solution with minimum $L^2(\Omega)$ norm. This approach for wavefront reconstruction was first suggested in [7].

Theorem 1. *The problem of computing the minimizer of the functional J_0 with minimum $L^2(\Omega)$ norm in $H_0^1(\Omega)$ is well-posed provided $\nabla \cdot g \in L^2(\Omega)$.*

Proof. First, we note that $\Delta\phi = f$ with homogeneous Neumann boundary conditions has weak solutions in $H^1(\Omega)$ provided $\int_\Omega f\,dx\,dy \in L^2(\Omega)$ [5]. Thus, it follows that since $\nabla \cdot g \in L^2(\Omega)$ and $\int_\Omega \nabla \cdot g\,dx\,dy = 0$ (Gauss' Theorem), weak solutions of

$$\begin{aligned} \nabla \cdot (\mathcal{M}\nabla\phi) &= \nabla \cdot g \quad \text{on} \quad \Omega, \\ \partial\phi/\partial\vec{n} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{15}$$

exist in $H^1(\Omega)$. They also happen to be minimizers of J_0. Since the $L^2(\Omega)$ norm is strongly convex and the minimizers of J_0 form a convex set, a minimizer of J_0 with minimum $L^2(\Omega)$ norm, which we will denote ϕ_{MNLS}, exists, is unique, and satisfies $\phi_{\mathrm{MNLS}} = 0$ on $\partial\Omega$. Thus $\phi_{\mathrm{MNLS}} \in H_0^1(\Omega)$.

Finally, since the nonzero eigenvalues of $\nabla \cdot \mathcal{M}\nabla$ are bounded away from zero, the minimum norm solution depends continuously on the data g, where perturbations in g must are measured with the $H^1(\Omega)$ norm. Thus the problem is well-posed as desired. □

Remark: We note that while the computation of the minimum norm least squares solution is well-posed, ϕ_{MNLS} is not guaranteed to lie in $H_0^2(\Omega)$, which we have deemed to be desirable.

Minimizers of J_σ are weak solutions of its Euler-Lagrange PDE. To obtain this PDE, we compute the first variation of J_σ. Suppose ϕ is an infinitely differentiable minimizer of (12) satisfying the boundary conditions $\phi = \partial\phi/\partial\vec{n} = 0$. Then, using integration by parts, we have

$$\begin{aligned} 0 &= \left.\frac{d}{d\tau} J_\sigma(\phi + \tau\psi)\right|_{\tau=0}, \\ &= 2\int_\Omega \langle (\mathcal{M}\nabla\phi - g), \nabla\psi\rangle\, dx + \frac{\sigma^2}{c_0}\int_\Omega \Delta\phi\Delta\psi\, dx, \\ &= 2\int_\Omega \psi\left(-\nabla\cdot(\mathcal{M}\nabla\phi) + \frac{\sigma^2}{c_0}\Delta^2\phi + \nabla\cdot g\right) dx\, dy, \end{aligned} \tag{16}$$

for all $\psi \in H_0^2(\Omega)$. Thus

$$-\nabla\cdot(\mathcal{M}\nabla\phi) + (\sigma^2/c_0)\Delta^2\phi = -\nabla\cdot g, \quad \text{on} \quad \Omega \tag{17}$$

is the Euler-Lagrange equation for (12). The operator Δ^2 is known as the biharmonic.

Our task now is to show that the problem of minimizing (12) on $H_0^2(\Omega)$ is well-posed for all $\sigma^2 > 0$; that is, for every $\sigma^2 > 0$, (12) has a unique minimizer in $H_0^2(\Omega)$ that depends continuously on the data g.

Theorem 2. *The problem of computing a minimizer for J_σ in $H_0^2(\Omega)$ is well-posed for $\sigma^2 > 0$.*

Proof. First, we show that there exists a unique minimizer of J_σ. A similar computation to that above yields

$$\frac{d^2}{d\xi d\tau} J_\sigma(\phi + \tau\psi + \xi\psi)\bigg|_{\tau,\xi=0} = 2\int_\Omega (\langle \mathcal{M}\nabla\psi, \nabla\psi\rangle + (\sigma^2/c_0)(\Delta\psi)^2)\, dx\, dy, \tag{18}$$

Since Δ has a trivial null-space on $H_0^2(\Omega)$ with eigenvalues bounded away from zero, the functional on the right-hand side in (18) is strictly positive on $H_0^2(\Omega)$ when $\sigma^2 > 0$. Thus J_σ is a strongly convex functional on $H_0^2(\Omega)$. Furthermore, $J_\sigma(\phi) \to \infty$ whenever $\|\phi\|_{H_0^2(\Omega)} \to \infty$, and hence, J_σ is also coercive on $H_0^2(\Omega)$. Existence and uniqueness of solutions then follows from the fact that strictly convex, coercive functions on a Hilbert space have a unique minimizer [11, Theorem 2.30].

The fact that this minimizer depends continuously on g follows immediately from an appeal to (17) together with the fact that the eigenvalues of the biharmonic operator Δ^2 are bounded away from zero on $H_0^2(\Omega)$. Note that since $\nabla \cdot g$ appears on the right-hand side in (17), changes in g must be measured using $H^1(\Omega)$ norm. □

Remark: Thus the minimum variance estimate has the desired smoothness properties.

It is also important to determine what the minimizers of J_σ converge to as $\sigma^2 \to 0^+$.

Theorem 3. *Let ϕ_σ be the minimizer of J_σ. Then as $\sigma^2 \to 0^+$, ϕ_σ converges to the weak solution of (15), i.e. the minimizer of J_0, that minimizes $\int_\Omega (\Delta\phi)^2\, dx\, dy$.*

Proof. Let ϕ_0 be a weak solution of (15) in $H_0^2(\Omega)$. Then $J_\sigma(\phi_0) \to J_0(\phi_0)$, and hence, $J_\sigma(\phi_0) \geq J_\sigma(\phi_\sigma) \geq J_0(\phi_0)$ implies $J_\sigma(\phi_\sigma) \to J_0(\phi_0)$. Now, using the above derivative computations, we expand J_σ in a Taylor series about ϕ_σ to obtain

$$\begin{aligned} J_\sigma(\phi_0) - J_\sigma(\phi_\sigma) &= J_\sigma(\phi_\sigma + (\phi_0 - \phi_\sigma)) - J_\sigma(\phi_\sigma), \\ &= \int_\Omega (\mathcal{M}\nabla(\phi_0 - \phi_\sigma))^2\, dx\, dy + \frac{\sigma}{c_0}\int_\Omega (\Delta(\phi_0 - \phi_\sigma))^2\, dx\, dy. \end{aligned}$$

Since $J_\sigma(\phi_0) - J_\sigma(\phi_\sigma)$ converges to zero as $\sigma \to 0^+$, we have

$$\mathcal{M}\nabla(\phi_0 - \phi_\sigma) \to 0,$$

and hence, ϕ_σ converges to a weak solution of (15), which we will denote ϕ^*.

We now show that ϕ^* is the weak solution of (15) minimizing $\|\Delta\phi\|_2^2$. For this, we consider the constrained problem

$$\min_\phi \frac{1}{2}\int_\Omega (\Delta\phi)^2\, dx\, dy \quad \text{s.t.} \quad \int_\Omega (\mathcal{M}\nabla\phi - g)^2\, dx\, dy \le C, \tag{19}$$

for $\phi \in H_0^2(\Omega)$, which has Karush-Kuhn-Tucker conditions with weak form

$$\int_\Omega (\Delta\phi)^2\, dx\, dy + \lambda \int_\Omega (\mathcal{M}\nabla\phi - g)^2\, dx\, dy \;=\; 0, \tag{20}$$

$$\lambda\left(\int_\Omega (\mathcal{M}\nabla\phi - g)^2\, dx\, dy - C\right) \;=\; 0, \tag{21}$$

and $\lambda > 0$. Since the objective function in (19) is strictly convex on $H_0^2(\Omega)$, and the constraint function is convex, if ϕ and λ satisfy (20), (21) then ϕ is the unique solution of (19). Finally, as $C \to 0$ in (21) it must be that $\lambda \to \infty$ in (20), which corresponds to $\sigma^2 \to 0^+$ in (12). Thus we have that as $\sigma^2 \to 0^+$, ϕ_σ converges to a solution of (19) with $C = 0$, which is what we wanted to show. □

Remarks:

1. Note that the least squares solution that minimizes $\int_\Omega (\Delta\phi)^2\, dx\, dy$ must live in $H_0^2(\Omega)$ and will therefore be a strong solution of (15). Thus for exact data this solution will coincide with ϕ_{true}.

2. Results analogous to Theorems 1, 2, and 3 can be obtained if in (8), C_ϕ is defined by (9), (10) instead. Note that C_ϕ is positive definite with eigenvalues bounded away from zero. The corresponding minimum variance estimates will then converge to the weak solution of (15) that minimizes the prior probability density (8) with C_ϕ defined by (9), (10).

We finish with an analysis of the GDLS method discussed in the introduction. Suppose that the hypothesis of Theorem 1 hold, and define $\phi^\sigma_{\text{GDLS}}$ to be the minimizer of (13).

Theorem 4. *The problem of computing $\phi^\sigma_{\text{GDLS}} \in H_0^2(\Omega)$ is well-posed. Furthermore, as $\sigma^2 \to 0^+$, $\phi^\sigma_{\text{GDLS}}$ converges to the weak solution of (15), i.e. the minimizer of J_0, that minimizes $\int_\Omega (\Delta\phi)^2\, dx\, dy$.*

Proof. We begin by noting that from [5, Problem 6.6.4] it follows that since $\phi_{\mathrm{MNLS}} \in H_0^1(\Omega)$, the minimizer of (13), which is unique, is also the strong solution of

$$(\sigma^2/c_0)\Delta\phi + \phi = \phi_{\mathrm{MNLS}}, \quad \partial\phi/\partial\vec{n} = 0, \tag{22}$$

and is therefore contained in $H_0^2(\Omega)$.

Since the eigenvalues of the operator $(\sigma^2/c_0)\Delta + \mathcal{I}$ are bigger than or equal to one, $\phi^{\sigma}_{\mathrm{GDLS}}$ depends continuously on ϕ_{MNLS}. Thus solving (22), or, equivalently, minimizing (13), is well-posed. The well-posedness of the computation of $\phi^{\sigma}_{\mathrm{GDLS}}$ then follows from the fact that the computation of ϕ_{MNLS} is well-posed, which we proved in Theorem 1.

Arguments analogous to those found in the proof of Theorem 3 yield the desired convergence result. □

Remark: The fact that GDLS solutions are contained in $H_0^2(\Omega)$ and converge to the same least squares solution when $\sigma \to 0^+$ as do the minimum variance solutions, makes it a desirable approach.

3 Conclusions

We have presented a theoretical analysis of several methods for wavefront reconstruction that are meant for use in adaptive optics (AO) systems in ground base astronomy. Our focus has been on the wavefront reconstruction problem in which the data consists of measurements of the wavefront gradient.

Several methods have been proposed for the solution of this problem. The classical approach involves solving a least squares problem, whereas a more stable and accurate approach results if a minimum variance, or penalized least squares, approach is taken instead. We also present, and analyze, a method in which the gradient denoised least squares solution is computed.

Our results show that each approach is well-posed, though in the least squares case, this required the computation of the solution with minimum $L^2(\Omega)$ norm. Well-posedness is important because then we know that these methods are stable with respect to measurement and modelling errors.

We also showed that both the minimum variance and GDLS solutions are contained in $H_0^2(\Omega)$, which is where we argued the true phase should lie. The least squares solutions, on the other hand, can only be guaranteed to lie in

$H_0^1(\Omega)$. This gives further motivation for the use of the minimum variance and GDLS methods.

Finally, we showed that as the parameter $\sigma^2 \to 0^+$, the minimum variance and GDLS solutions converge to the same least squares solution. Given the proven effectiveness of minimum variance estimation together with the fact that GDLS yields very computationally efficient estimation schemes, this suggests that GDLS is an approach worthy of further consideration for use in operational AO.

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