

## An Analysis of Regularization by Diffusion for Ill-Posed Poisson Likelihood Estimation

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The noise contained in images collected by a charge coupled device (CCD) camera is predominantly of Poisson type. This motivates the use of the negative-log of the Poisson likelihood function in place of the ubiquitous least squares fit-to-data. However if the underlying mathematical model is assumed to have the form  $z = Au$ , where  $z$  is the data and  $A$  is a linear, compact operator, minimizing the negative-log of the Poisson likelihood function is an ill-posed problem, and hence some form of regularization is required. In previous work, the authors have performed theoretical analyses of two approaches for regularization in this setting: standard Tikhonov regularization in [2] and total variation regularization in [3]. In this paper, we consider a class of regularization functionals defined by differential operators of diffusion type, and our main results constitute a theoretical justification of this approach. However, in order to demonstrate that the approach is effective in practice, we follow our theoretical analysis with a numerical experiment.

### 1 Introduction

We consider the problem of solving

$$z(x) = Au(x) \stackrel{\text{def}}{=} \int_{\Omega} a(x; y)u(y) dy, \quad (1)$$

for  $u$  on a closed bounded domain  $\Omega \subset \mathbb{R}^n$ . We assume  $a \in L^2(\Omega \times \Omega)$  so that  $A : L^p(\Omega) \rightarrow L^2(\Omega)$  is a compact, positive semi-definite linear operator. We also assume that  $Au \geq 0$  whenever  $u \geq 0$  and that the data  $z \geq 0$  is contained in  $L^\infty(\Omega)$ . Then the problem of solving (1) is ill-posed [16, 17]. Finally, we denote the true solution by  $u_{\text{exact}} \geq 0$  and assume that it is a solution of (1), where  $A$  and  $z$  are the error free operator and data respectively.

Since in practice the exact data and operator can only be estimated, we want our solution method to have the property that small errors in our measurements of  $z$  and  $A$  will result in correspondingly small changes in the estimate

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of  $u_{\text{exact}}$  obtained by our method. Due to the ill-posedness of (1) regularization is needed in order for this property to be satisfied.

For us, this will require a variational formulation of the problem. The most common example of a variational formulation of (1) is the least squares problem

$$\arg \min_u \|Au - z\|_2^2, \quad (2)$$

where “arg min” means “argument of the minimum”. Given that  $A$  is compact, (2) is also an ill-posed problem. Regularization techniques for (2) have been extensively studied [16, 17].

The variational problem that is of interest to us in this paper arises from statistical considerations regarding the noise in typical astronomical and medical imaging data (see [3, 15] for details). It is given by

$$\arg \min_{u \geq 0} T_0(Au; z + \gamma), \quad (3)$$

where

$$T_0(Au; z + \gamma) \stackrel{\text{def}}{=} \int_{\Omega} ((Au + \gamma + \sigma^2) - (z + \gamma + \sigma^2) \log(Au + \gamma + \sigma^2)) \, dx. \quad (4)$$

Here the positive parameters  $\gamma$  and  $\sigma^2$  are defined by the statistical noise model for CCD camera noise given in [15], and are known, at least approximately, in practice.

To see that  $u_{\text{exact}}$  is a minimizer of  $T_0$ , we compute the gradient and Hessian of  $T_0$  with respect to  $u$ , which are given, respectively, by

$$\nabla T_0(Au; z + \gamma) = A^* \left( \frac{Au - z}{Au + \gamma + \sigma^2} \right), \quad (5)$$

$$\nabla^2 T_0(Au; z + \gamma) = A^* \left( \text{diag} \left( \frac{z + \gamma + \sigma^2}{(Au + \gamma + \sigma^2)^2} \right) \right) A, \quad (6)$$

where “ $*$ ” denotes operator adjoint, and  $\text{diag}(v)$  is the linear operator defined by  $\text{diag}(v)w \stackrel{\text{def}}{=} vw$ . Since  $z \geq 0$ ,  $Au \geq 0$  for  $u \geq 0$ , and  $A$  is positive semi-definite (see comments following (1)),  $\nabla^2 T_0$  is a positive semi-definite operator for all  $u \geq 0$ . Thus  $T_0$  is convex [17, Theorem 2.42] with minimizer  $u_{\text{exact}}$  as desired. If  $A$  is positive definite,  $u_{\text{exact}}$  is the unique minimizer of  $T_0$ .

As in the least squares case, since  $A$  is compact, the problem of minimizing  $T_0(u)$  is ill-posed and must be regularized [16,17]. For us, this involves replacing  $T_0(Au; z + \gamma)$  by the parameter dependent functional

$$T_\alpha(Au; z + \gamma) \stackrel{\text{def}}{=} T_0(Au; z + \gamma) + \alpha J(u), \quad (7)$$

where  $\alpha > 0$  is the regularization parameter and  $J$  the regularization functional. The corresponding variational problem of interest is then

$$u_\alpha \stackrel{\text{def}}{=} \arg \min_{u \in \mathcal{C}} T_\alpha(Au; z + \gamma), \quad (8)$$

where  $\mathcal{C}$  consists of the nonnegative subset of a function space determined by the regularization functional  $J$ . We restrict ourselves to  $\mathcal{C}$  because the values of  $u$  correspond to intensities of the unknown image, which are nonnegative. Moreover,  $T_\alpha$  is defined on this set, but not necessarily on all of  $\mathbb{R}^n$ .

In previous work of the authors, a theoretical analysis of (8) is performed when  $J(u) = \|u\|_2^2$  [2] and when  $J$  is the total variation function [3], that is when  $J(u) \stackrel{\text{def}}{=} \int_\Omega \sqrt{|\nabla u|^2} dx$ . In both of these papers, the associated computational problem is also considered. Moreover, in [1], the computational problem for the total variation regularization case, which is significantly more challenging, is further considered; specifically, a highly efficient method is introduced and its convergence is proved. We adapt the same method for use on the computational problem considered here.

In this paper, we will study the use of a class of quadratic regularization functions defined by differential operators of diffusion type. In the next section, we define the regularization functionals that are of interest to us and present mathematical preliminaries that will be needed in the theoretical analysis that is the topic of Section 2. In Section 3, we shift gears and focus on the computational problem that must be solved in practice, and end with conclusions in Section 4.

## 2 Theoretical Analysis

### 2.1 Objectives

Our main objective is to provide a theoretical justification for using (8). This involves proving that solutions of (8) exist and depend continuously on  $A$  and  $z$ ; that is, given a sequence of perturbed compact operator equations

$$z_n(x) = A_n u(x) \stackrel{\text{def}}{=} \int_\Omega a_n(x; y) u(y) dy, \quad (9)$$

where  $A_n$  and  $z_n$  satisfy the same conditions as  $A$  and  $z$  (outlined following (1)), and solutions  $u_{\alpha,n}$  of the corresponding minimization problems

$$u_{\alpha,n} \stackrel{\text{def}}{=} \arg \min_{u \in \mathcal{C}} T_\alpha(A_n u; z_n), \quad (10)$$

$u_{\alpha,n} \rightarrow u_\alpha$  provided  $A_n \rightarrow A$  and  $z_n \rightarrow z + \gamma$ .

We also must show that a sequence of positive regularization parameters  $\{\alpha_n\}$  can be chosen so that  $u_{\alpha_n,n} \rightarrow u_{\text{exact}}$  as  $\alpha_n \rightarrow 0$ , where  $u_{\alpha_n,n}$  is the minimizer of  $T_{\alpha_n}(A_n u; z_n)$  over  $\mathcal{C}$ .

We note that these two convergence results are not only of academic interest, since in practice one always deals with some perturbed approximation of an (assumed) exact underlying model.

## 2.2 Preliminaries

In this section, we define the regularization functionals that we will use. Let

$$\langle u, v \rangle_{H^1(\Omega)} \stackrel{\text{def}}{=} \langle u, v \rangle_2 + \langle \nabla u, \nabla v \rangle_2, \quad (11)$$

where “ $\nabla$ ” denotes the gradient. The set of all functions  $u \in C^1(\Omega)$  for which

$$\|u\|_{H^1(\Omega)} = \sqrt{\langle u, u \rangle_{H^1(\Omega)}}$$

is finite is a normed linear space whose closure in  $L^2(\Omega)$  is the Sobolev space  $H^1(\Omega)$  [13]. We note, moreover, that with the inner-product defined in (11),  $H^1(\Omega)$  is a Hilbert space. We can now define  $\mathcal{C}$ :

$$\mathcal{C} = \{u \in H^1(\Omega) \mid u \geq 0\}.$$

Now, we define the  $d \times d$  matrix valued function

$$[\mathbf{\Lambda}(x)]_{ij} = \lambda_{ij}(x), \quad 1 \leq i, j \leq d,$$

where  $\mathbf{\Lambda}(x)$  is non-singular for all  $x \in \bar{\Omega}$ . Moreover, we assume that the  $\lambda_{ij}$ 's are continuously differentiable for all  $i$  and  $j$ . We then define our regularization functional, as in [11], by

$$J(u) = \|\mathbf{\Lambda}(x)\nabla u\|_2^2. \quad (12)$$

Assuming homogeneous Dirichlet boundary conditions, as we will later on, (12) can be written, using integration by parts [8], as

$$J(u) = \int_{\Omega} u(-\nabla \cdot (\bar{\Lambda}(x)\nabla u)) dx,$$

where  $\bar{\Lambda} = \Lambda^T \Lambda$  and  $u$  is assumed to be twice differentiable. Hence, we see that (12) is the functional to be minimized in a variational formulation of the diffusion equation  $-\nabla \cdot (\bar{\Lambda}(x)\nabla u) = 0$ , subject to  $u = 0$  on  $\partial\Omega$ .

With (8), (7), (4), (12) (denoted only by (8) in the sequel) we have defined our variational problem. We can now present the main results of the paper.

### 2.3 Continuous Dependence of Minimizers on Perturbations in $A$ and $z$

In order to simplify the notation in our arguments, we will use  $T_{\alpha}(u)$  to denote  $T_{\alpha}(Au; z + \gamma)$  and  $T_{\alpha,n}(u)$  to denote  $T_{\alpha}(A_n u; z_n)$  throughout the remainder of the paper.

Before continuing, we need the following definition: we will say that a function  $T$  is *coercive* if

$$T(u) \rightarrow +\infty \quad \text{whenever} \quad \|u\|_{H^1(\Omega)} \rightarrow +\infty. \quad (13)$$

In order to prove the existence and uniqueness of solutions of (8), we will use the following theorem, which is similar to [17, Theorem 2.30].

**THEOREM 2.1** *If  $T : H^1(\Omega) \rightarrow \mathbb{R}$  is convex and coercive, then it has a minimizer on  $\mathcal{C}$ . If  $A$  is positive definite,  $T$  is strictly convex and hence has a unique minimizer over  $\mathcal{C}$ .*

*Proof* Let  $\{u_n\} \subset \mathcal{C}$  be such that  $T(u_n) \rightarrow T_* \stackrel{\text{def}}{=} \inf_{u \in \mathcal{C}} T(u)$ . Then, by (13), the sequence  $\{u_n\}$  is bounded in  $H^1(\Omega)$ . By the Rellich-Kondrachov Compactness Theorem [8, Theorem 1, Section 5.7], this implies that  $\{u_n\}$  has a subsequence  $\{u_{n_j}\}$  that converges to some  $u_* \in \mathcal{C}$ . Now, since  $T$  is convex, it is weakly lower semi-continuous [18], and hence,

$$T(u_*) \leq \liminf T(u_{n_j}) = \lim T(u_n) = T_*.$$

Thus  $u_*$  minimizes  $T$  on  $\mathcal{C}$ . Uniqueness follows immediately if  $T$  strictly convex.

□

**COROLLARY 2.2** *(Existence and Uniqueness of Minimizers)  $T_{\alpha}$  has a minimizer over  $\mathcal{C}$ . This minimizer is unique if the operator  $A$  is positive definite.*

*Proof* First, recall that  $T_0$  is a convex functional. Next, note that

$$\|\mathbf{\Lambda}(x)\nabla u\|^2 = \sum_{\ell=1}^d \int_{\Omega} \left( \sum_{j=1}^d a_{\ell j}(x) \partial_j u \right)^2 dx \quad (14)$$

$$= \sum_{\ell=1}^d \int_{\Omega} \bar{a}_{\ell k}(x) (\partial_j u) (\partial_k u) dx \quad (15)$$

where  $\bar{\lambda}_{jk}(x) \stackrel{\text{def}}{=} \sum_{\ell=1}^d \lambda_{\ell j}(x) \lambda_{\ell k}(x)$ . It is readily seen that if  $[\bar{A}(x)]_{jk} \stackrel{\text{def}}{=} \bar{a}_{jk}(x)$ , then  $\bar{\mathbf{\Lambda}}(x) = \mathbf{\Lambda}^T(x)\mathbf{\Lambda}(x)$ , and hence  $\bar{\mathbf{\Lambda}}(x)$  is symmetric and positive definite for all  $x$ . Since  $\bar{\Omega}$  is a closed and bounded set, the eigenvalues of  $\bar{\mathbf{\Lambda}}(x)$  will be uniformly bounded away from 0 on  $\Omega$ . Thus we have that there exists a constant  $\theta > 0$  such that

$$\sum_{j,k=1}^n \bar{\lambda}_{jk}(x) \xi_j \xi_k \geq \theta |\boldsymbol{\xi}|^2 \quad (16)$$

for all  $x \in \Omega$  and  $\boldsymbol{\xi} \in \mathbb{R}^n$ . Equation (16) implies that for  $u \in H^1(\Omega)$ ,

$$J(u) \geq \theta \int_{\Omega} \nabla u \cdot \nabla u dx. \quad (17)$$

Moreover, by the Poincaré inequality [8, Theorem 3, Chapter 5], we have that there exists a constant  $C$  such that

$$\|u\|_2^2 \leq C \|\nabla u\|_2^2.$$

Therefore,

$$J(u) + C\theta \|\nabla u\|_2^2 \geq \theta \|u\|_{H^1(\Omega)}^2.$$

But by (17),  $\|\nabla u\|_2^2 \leq \theta^{-1} J(u)$ , and so we have, finally,

$$J(u) \geq \frac{\theta}{1+C} \|u\|_{H^1(\Omega)}^2. \quad (18)$$

Thus  $J$  is coercive, and is convex by results shown in [7].

The convexity of  $T_{\alpha}$  over  $\mathcal{C}$  follows immediately, with strict convexity if  $A$  is positive definite.

To show that  $T_\alpha$  is coercive, we note that by Jensen's inequality and the properties of the function  $x - c \log x$  for  $c > 0$ ,

$$\begin{aligned} T_0(u) &\geq \|Au + \gamma + \sigma^2\|_1 - \|z + \gamma + \sigma^2\|_\infty \log \|Au + \gamma + \sigma^2\|_1, \\ &\geq \|z + \gamma + \sigma^2\|_\infty - \|z + \gamma + \sigma^2\|_\infty \log \|z + \gamma + \sigma^2\|_\infty. \end{aligned} \quad (19)$$

Since  $z \geq 0$ ,  $T_0$  is bounded below. Thus the coercivity of  $J$  implies the coercivity of  $T_\alpha$ .

The desired result then follows from Theorem 2.1  $\square$

**Remark:** The choice of boundary conditions will also determine the space in which solutions can be expected to be found. For example, if the boundary condition  $u = 0$  on  $\partial\Omega$  is used, then  $J$  will be strictly convex on the set  $H_0^1(\Omega) \stackrel{\text{def}}{=} \{u \in H^1(\Omega) \mid u = 0\}$  [8], and hence, by the arguments above  $T_\alpha$  will have a unique minimizer on  $\mathcal{C}_0 = \{u \in H_0^1(\Omega) \mid u \geq 0\}$  even if  $A$  is only positive semi-definite.

Recalling that  $A_n$  and  $z_n$  in (9) satisfy the same assumptions as  $A$  and  $z$ , we have from the above arguments that  $u_{\alpha,n}$  (defined in (10)) exists, and is unique if  $A_n$  is invertible.

To prove stability, we assume that  $T_\alpha$  has a unique minimizer  $u_\alpha$  over  $\mathcal{C}$  and then show that  $A_n \rightarrow A$  and  $z_n \rightarrow z + \gamma$  implies  $u_{\alpha,n} \rightarrow u_\alpha$ , even if the  $A_n$  are only positive semi-definite. The following theorem gives conditions that guarantee this result. For completeness, we present the proof.

**THEOREM 2.3** *Assume that  $A$  is positive definite, and let  $u_\alpha$  be the unique minimizer of  $T_\alpha$  over  $\mathcal{C}$ . Also, for each  $n \in \mathbb{N}$  let  $u_{\alpha,n}$  a minimizer of  $T_{\alpha,n}$  over  $\mathcal{C}$ . Suppose, furthermore, that*

(i) *for any sequence  $\{u_n\} \subset H^1(\Omega)$ ,*

$$\lim_{n \rightarrow \infty} T_{\alpha,n}(u_n) = +\infty \quad \text{whenever} \quad \lim_{n \rightarrow \infty} \|u_n\|_{H^1(\Omega)} = +\infty; \quad (20)$$

(ii) *given  $B > 0$  and  $\epsilon > 0$ , there exists  $N$  such that*

$$|T_{\alpha,n}(u) - T_\alpha(u)| < \epsilon \quad \text{whenever} \quad n \geq N, \quad \|u\|_{H^1(\Omega)} \leq B. \quad (21)$$

*Then*

$$\lim_{n \rightarrow \infty} \|u_{\alpha,n} - u_\alpha\|_2 = 0. \quad (22)$$

*Proof* Note that  $T_{\alpha,n}(u_{\alpha,n}) \leq T_{\alpha,n}(u_\alpha)$ . From this and (21), we have

$$\liminf T_{\alpha,n}(u_{\alpha,n}) \leq \limsup T_{\alpha,n}(u_{\alpha,n}) \leq T_\alpha(u_\alpha) < \infty. \quad (23)$$

Thus by (20), the  $u_{\alpha,n}$ 's are bounded in  $H^1(\Omega)$ . The Rellich-Kondrachev Compactness Theorem [8, Theorem 1, Section 5.7] then tells us that there exists a subsequence  $\{u_{\alpha,n_j}\}$  that converges to some  $\hat{u} \in L^2(\Omega)$ . Furthermore, by the weak lower semicontinuity of  $T_\alpha$ , (21), and (23) we have

$$\begin{aligned} T_\alpha(\hat{u}) &\leq \liminf T_\alpha(u_{\alpha,n_j}), \\ &= \liminf (T_\alpha(u_{\alpha,n_j}) - T_{\alpha,n_j}(u_{\alpha,n_j})) + \liminf T_{\alpha,n_j}(u_{\alpha,n_j}), \\ &\leq T_\alpha(u_\alpha). \end{aligned}$$

By uniqueness of minimizers,  $\hat{u} = u_\alpha$ . Thus every convergent subsequence of  $\{u_{\alpha,n}\}$  converges to  $u_\alpha$ , and hence, we have (22).  $\square$

The following corollary of Theorem 2.3 is the stability result for (8) that we seek.

**COROLLARY 2.4** (*Stability of Minimizers*) *Assume  $\|z_n - (z + \gamma)\|_\infty \rightarrow 0$ , and that  $A_n \rightarrow A$  in the  $L^1(\Omega)$  operator norm, where  $A$  is positive definite. Then*

$$\lim_{n \rightarrow \infty} \|u_{\alpha,n} - u_\alpha\|_2 = 0.$$

*Proof* It suffices to show that conditions (i) and (ii) from Theorem 2.3 hold. For condition (i), note that the analogue of inequality (19) for  $T_{0,n}$  is given by

$$T_{0,n}(u_{\alpha,n}) \geq \|z_n + \sigma^2\|_\infty - \|z_n + \sigma^2\|_\infty \log \|z_n + \sigma^2\|_\infty,$$

which has a lower bound for all  $n$  since  $\|z_n - (z + \gamma)\|_\infty \rightarrow 0$  and  $z \in L^\infty(\Omega)$  is nonnegative. Thus, by (18),  $T_{\alpha,n}(u_{\alpha,n}) = T_{0,n}(u_{\alpha,n}) + \alpha J(u_{\alpha,n}) \rightarrow +\infty$  whenever  $\|u_{\alpha,n}\|_{H^1(\Omega)} \rightarrow \infty$ , and hence (20) is satisfied.

For condition (ii), note that using Jensen's inequality and the properties of

the logarithm

$$\begin{aligned}
|T_{\alpha,n}(u) - T_\alpha(u)| &= \left| \int_{\Omega} ((A_n - A)u - (z_n + \sigma^2) \log(A_n u + \gamma + \sigma^2)) \, dx \right. \\
&\quad \left. + \int_{\Omega} ((z + \gamma + \sigma^2) \log(Au + \gamma + \sigma^2)) \, dx \right|, \\
&\leq \|A_n - A\|_1 \|u\|_1 \\
&\quad + \|z_n - (z + \gamma)\|_\infty \log(\|A_n\|_1 \|u\|_1 + (\gamma + \sigma^2)|\Omega|) \quad (24) \\
&\quad + \|z + \gamma + \sigma^2\|_\infty \log \left\| \frac{(Au + \gamma + \sigma^2)}{(A_n u + \gamma + \sigma^2)} \right\|_1.
\end{aligned}$$

By assumption,  $\|A_n - A\|_1, \|z_n - (z + \gamma)\|_\infty \rightarrow 0$ . Furthermore, by the Banach-Steinhaus Theorem,  $\|A_n\|_1$  is uniformly bounded. Since we are assuming that  $\|u\|_{H^1(\Omega)}$  is bounded, we know have that  $\|u\|_2$  will be bounded as well. Moreover, since  $\Omega$  is a bounded set, this implies that  $\|u\|_1$  is bounded. Thus the first two terms on the right-hand side in (24) tend to zero as  $n \rightarrow \infty$ . For the third term note that

$$\left\| \frac{Au + \gamma + \sigma^2}{A_n u + \gamma + \sigma^2} - 1 \right\|_1 \leq \left\| \frac{1}{A_n u + \gamma + \sigma^2} \right\|_1 \|A_n - A\|_1 \|u\|_1,$$

which converges to zero since  $\|1/(A_n u + \gamma + \sigma^2)\|_1$  is bounded and  $\|A_n - A\|_1 \rightarrow 0$ . Thus  $\log(\|(Au + \gamma + \sigma^2)/(A_n u + \gamma + \sigma^2)\|_1) \rightarrow \log(1) = 0$ , and hence

$$|T_{\alpha,n}(u) - T_\alpha(u)| \rightarrow 0. \quad (25)$$

The desired result now follows from Theorem 2.3.  $\square$

## 2.4 Convergence of Minimizers

It remains to show that a sequence of positive regularization parameters  $\{\alpha_n\}$  can be chosen so that  $u_{\alpha_n,n} \rightarrow u_{\text{exact}}$  as  $\alpha_n \rightarrow 0$ . For this, we assume that  $A$  is positive definite and hence that  $u_{\text{exact}}$  is the unique solution of (1). However, we assume that the  $A_n$ 's are only positive semi-definite.

**THEOREM 2.5 (Convergence of Minimizers)** *Suppose  $\|z_n - (z + \gamma)\|_\infty \rightarrow 0$ ,  $A_n \rightarrow A$  in the  $L^1(\Omega)$  operator norm with  $A$  positive definite, and that  $\alpha_n \rightarrow 0$  at a rate such that*

$$(T_{0,n}(u_{\text{exact}}) - T_{0,n}(u_{0,n})) / \alpha_n \quad (26)$$

*is bounded. Then  $u_{\alpha_n,n} \rightarrow u_{\text{exact}}$  strongly in  $L^2(\Omega)$ .*

*Proof* Since  $u_{\alpha_n, n}$  minimizes  $T_{\alpha_n, n}$ , we have

$$T_{\alpha_n, n}(u_{\alpha_n, n}) \leq T_{\alpha_n, n}(u_{\text{exact}}). \quad (27)$$

Since  $\{z_n\}$  and  $\{A_n\}$  are uniformly bounded and  $A_n \rightarrow A$  in the  $L^1(\Omega)$  operator norm,  $\{T_{\alpha_n, n}(u_{\text{exact}})\}$  is a bounded sequence. Hence  $\{T_{\alpha_n, n}(u_{\alpha_n, n})\}$  is bounded by (27).

Subtracting  $T_{0, n}(u_{0, n})$  from both sides of (27) and dividing by  $\alpha_n$  yields

$$\begin{aligned} (T_{0, n}(u_{\alpha_n, n}) - T_{0, n}(u_{0, n}))/\alpha_n + \alpha J(u_{\alpha_n, n}) &\leq (T_{0, n}(u_{\text{exact}}) - T_{0, n}(u_{0, n}))/\alpha_n \\ &\quad + \alpha J(u_{\text{exact}}). \end{aligned} \quad (28)$$

By (26), the right-hand side is bounded, implying the left hand side is bounded. Since  $T_{0, n}(u_{\alpha_n, n}) - T_{0, n}(u_{0, n})$  is nonnegative, this implies that  $\{J(u_{\alpha_n, n})\}$  is bounded. Equation (18) then tells us that  $\{u_{\alpha_n, n}\}$  is bounded in  $H^1(\Omega)$ .

We now show that  $u_{\alpha_n, n} \rightarrow u_{\text{exact}}$  in  $H^1(\Omega)$  by showing that every subsequence of  $\{u_{\alpha_n, n}\}$  contains a subsequence that converges to  $u_{\text{exact}}$ . Since  $\{u_{\alpha_n, n}\}$  is bounded in  $H^1(\Omega)$ , by the Rellich-Kondrachov Compactness Theorem [8, Theorem 1, Section 5.7] each of its subsequences in turn has a subsequence that converges strongly in  $L^2(\Omega)$ . Let  $\{u_{\alpha_{n_j}, n_j}\}$  be such a sequence and  $\hat{u}$  its limit. Then

$$\begin{aligned} T_0(\hat{u}) &= \int_{\Omega} (A(\hat{u} - u_{\alpha_{n_j}, n_j}) + (A - A_{n_j})u_{\alpha_{n_j}, n_j}) \, dx \, dy \\ &\quad + \int_{\Omega} (z_{n_j} - (z + \gamma)) \log(A\hat{u} + \gamma + \sigma^2) \, dx \, dy \\ &\quad - \int_{\Omega} (z_{n_j} + \sigma^2) \log((A_{n_j}u_{\alpha_{n_j}, n_j} + \gamma + \sigma^2)/(A\hat{u} + \gamma + \sigma^2)) \, dx \, dy \\ &\quad + T_{0, n_j}(u_{\alpha_{n_j}, n_j}), \end{aligned}$$

which, as in previous arguments, yields

$$\begin{aligned} |T_{0, n_j}(u_{\alpha_{n_j}, n_j}) - T_0(\hat{u})| &\leq \int_{\Omega} A(\hat{u} - u_{\alpha_{n_j}, n_j}) \, dx \, dy \\ &\quad + \|z_{n_j} - (z + \gamma)\|_{\infty} \log(\|A\|_1 \|\hat{u}\|_1 + \gamma|\Omega|) \\ &\quad + \|z_{n_j} + \sigma^2\|_{\infty} \log \|(A_{n_j}u_{\alpha_{n_j}, n_j} + \gamma + \sigma^2)/(A\hat{u} + \gamma + \sigma^2)\|_1 \\ &\quad + \|A - A_{n_j}\|_1 \|u_{\alpha_{n_j}, n_j}\|_1. \end{aligned}$$

Then

$$\|z_{n_j} - (z + \gamma)\|_\infty \log(\|A\|_1 \|\hat{u}\|_1 + (\gamma + \sigma^2)|\Omega|) \rightarrow 0,$$

since  $\|z_{n_j} - (z + \gamma)\|_\infty \rightarrow 0$  and  $\log(\|A\|_1 \|\hat{u}\|_1 + (\gamma + \sigma^2)|\Omega|)$  is constant, and

$$\|A - A_{n_j}\|_1 \|u_{\alpha_{n_j}, n_j}\|_1 \rightarrow 0$$

since  $\|A - A_{n_j}\|_1 \rightarrow 0$ , and  $\|u_{\alpha_{n_j}, n_j}\|_1$  is bounded since  $\|u_{\alpha_{n_j}, n_j}\|_{H^1(\Omega)}$  is bounded and  $H^1(\Omega)$  is compactly embedded in  $L^2(\Omega) \subset L^1(\Omega)$ .

We know that  $A$  is a bounded linear operator and  $\Omega$  is a set of finite measure, therefore  $F(u) = \int_\Omega Au \, dx$  is a bounded linear functional on  $L^2(\Omega)$ . The convergence of  $\{u_{\alpha_{n_j}, n_j}\}$  then implies  $\int_\Omega Au_{\alpha_{n_j}, n_j} \, dx \rightarrow \int_\Omega A\hat{u} \, dx$ , which yields  $\int_\Omega A(\hat{u} - u_{\alpha_{n_j}, n_j}) \, dx \rightarrow 0$ .

Since  $A$  is compact, it is completely continuous, i.e.  $u_{\alpha_{n_j}, n_j} \rightarrow \hat{u}$  implies that  $\|Au_{\alpha_{n_j}, n_j} - A\hat{u}\|_1 \rightarrow 0$  (cf. [6, Prop. 3.3]). Thus, since  $\left\| \frac{1}{A\hat{u} + \gamma + \sigma^2} \right\|_1$  is bounded and

$$\begin{aligned} \left\| \frac{A_{n_j} u_{\alpha_{n_j}, n_j} + \gamma + \sigma^2}{A\hat{u} + \gamma + \sigma^2} - 1 \right\|_1 &\leq \left\| \frac{1}{A\hat{u} + \gamma + \sigma^2} \right\|_1 \|A_{n_j} u_{\alpha_{n_j}, n_j} - A\hat{u}\|_1, \\ &\leq \left\| \frac{1}{A\hat{u} + \gamma + \sigma^2} \right\|_1 \times \\ &\quad \left( \|A_{n_j} - A\|_1 \|u_{\alpha_{n_j}, n_j}\|_1 + \|Au_{\alpha_{n_j}, n_j} - A\hat{u}\|_1 \right), \end{aligned}$$

we have that  $\|z_{n_j} + \sigma^2\|_\infty \log \|(A_{n_j} u_{\alpha_{n_j}, n_j} + \gamma + \sigma^2)/(A\hat{u} + \gamma + \sigma^2)\|_1 \rightarrow 0$ . Therefore

$$T_0(\hat{u}) = \lim_{n_j \rightarrow \infty} T_{0, n_j}(u_{\alpha_{n_j}, n_j}).$$

Invoking (28), (26), and (25), respectively, yields

$$\lim_{n_j \rightarrow \infty} T_{0, n_j}(u_{\alpha_{n_j}, n_j}) = \lim_{n_j \rightarrow \infty} T_{0, n_j}(u_{\text{exact}}) = T_0(u_{\text{exact}}).$$

Thus  $T_0(\hat{u}) = T_0(u_{\text{exact}})$ . Since  $u_{\text{exact}}$  is the unique minimizer of  $T_0$ , we have  $\hat{u} = u_{\text{exact}}$ . Therefore  $\{u_{\alpha_{n_j}, n_j}\}$  converges strongly to  $u_{\text{exact}}$  in  $L^2(\Omega)$ .  $\square$

### 3 A Numerical Experiment

In this section, we demonstrate that the approach considered in the theoretical arguments above is actually effective in practice. First, we note that discretizing (8) yields the discrete version of (8) given by

$$\arg \min_{\mathbf{u} \in \mathcal{C}} \left\{ T_\alpha(\mathbf{u}) \stackrel{\text{def}}{=} T_0(\mathbf{u}) + \frac{\alpha}{2} \mathbf{u}^T \mathbf{L} \mathbf{u} \right\}, \quad (29)$$

where

$$T_0(\mathbf{u}) \stackrel{\text{def}}{=} \sum_{i=1}^{N^2} ([\mathbf{A}\mathbf{u}]_i + \gamma + \sigma^2) - \sum_{i=1}^{N^2} (z_i + \sigma^2) \log([\mathbf{A}\mathbf{u}]_i + \gamma + \sigma^2) \quad (30)$$

is obtained from  $T_0(u)$  using mid-point quadrature, and  $\mathbf{L}$  is a discretization of the differential operator “ $-\nabla \cdot (\bar{\mathbf{\Lambda}}(x) \nabla)$ ”, which is positive semidefinite due to our assumptions above regarding the matrix valued function  $\mathbf{\Lambda}(x)$  in (12). We note that we assume  $u = 0$  on the boundary of  $\Omega$ , so that  $J(u) = \int_{\Omega} u(-\nabla \cdot (\bar{\mathbf{\Lambda}}(x) \nabla u)) dx$ . Taking  $\mathbf{\Lambda}(x) = \mathbf{I}_{d \times d}$  in (12) then yields  $J(u) = \langle u, -\nabla^2 u \rangle$ , where “ $-\nabla^2$ ” is the negative Laplacian operator, which we discretize in the standard way to obtain  $\mathbf{L}$  and the corresponding regularization function  $J(\mathbf{u}) = \mathbf{u}^T \mathbf{L} \mathbf{u}$ . The assumption of zero boundary conditions will also have an effect in our discretization of  $T_0$ . In particular, the blurring matrix  $\mathbf{A}$  will then be block Toeplitz with Toeplitz blocks [10]. We assume in addition that it is positive semi-definite. Also  $\mathbf{z}$  is assumed to contain random noise. In fact, (30) is the likelihood function that arises from the CCD camera noise model found in [15], and it can be used to derive the Richardson-Lucy algorithm [17], which is popular in both astronomical and medical imaging.

To numerically solve (29), we apply the gradient projection–reduced Newton–conjugate gradient (GPRNCG) iterative method of [4]. The adaptation of GPRNCG to this problem is straightforward (in [4],  $\mathbf{L} = \mathbf{I}$  in (29)). We note that this iterative method was also effectively adapted for the case of total variation regularization in [1]. To save space, we do not give a detailed description of the algorithm here.

Before presenting our experiment, we note that our theoretical arguments also hold for the regularization approach presented in [5, 11]. However, a complete implementation of that methodology is sufficiently complicated that we feel that it warrants a separate paper.

We perform tests using the  $64 \times 64$  simulated satellite seen on the left-hand side in Figure 1. Generating corresponding blurred noisy data requires

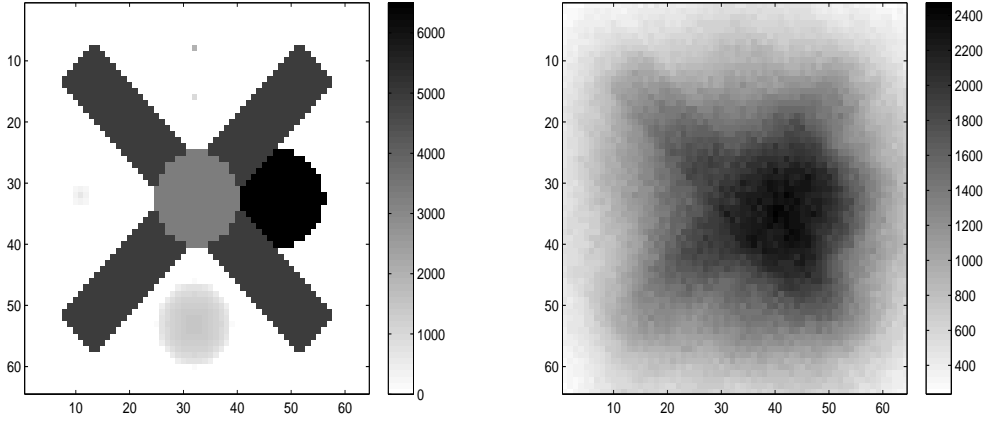


Figure 1. *On the left is the true object  $\mathbf{u}_{\text{true}}$ . On the right, is the blurred, noisy image  $\mathbf{z}$ .*

a discrete PSF  $\mathbf{a}$ , which we compute using the Fourier optics [9] PSF model

$$\mathbf{a} = \left| \text{fft2} \left( \mathbf{p} \odot e^{\hat{i}\phi} \right) \right|^2,$$

where  $\mathbf{p}$  is the  $N \times N$  indicator array for the telescopes pupil; “ $\odot$ ” denotes Hadamard (component-wise) product;  $\phi$  is the  $N \times N$  array that represents the aberrations in the incoming wavefronts of light;  $\hat{i} = \sqrt{-1}$ ; and  $\text{fft2}$  denotes the two-dimensional discrete Fourier transform. The  $64^2 \times 64^2$  blurring matrix  $\mathbf{A}$  is then defined by

$$\mathbf{A}\mathbf{u} = \text{ifft2}(\hat{\mathbf{a}} \odot (\text{fft2}(\mathbf{u}))), \quad \hat{\mathbf{a}} = \text{fft2}(\text{fftshift}(\mathbf{a})),$$

where  $\text{ifft2}$  is the inverse discrete Fourier transform and  $\text{fftshift}$  swaps the first and third and the second and fourth quadrants of the array  $\mathbf{a}$ . Then  $\mathbf{A}$  is block Toeplitz with Toeplitz blocks (BTTB) [10,17]. For efficient computations,  $\mathbf{A}$  is embedded in a  $128^2 \times 128^2$  block circulant with circulant block (BCCB) matrix, which can be diagonalized by the two-dimensional discrete Fourier and inverse discrete Fourier transform matrices [17]. Data  $\mathbf{z}$  with a signal-to-noise ratio of approximately 35 is then generated using the statistical model

$$\hat{\mathbf{z}} \sim \text{Poiss}(\mathbf{A}\mathbf{u}_{\text{exact}}) + \text{Poiss}(\gamma \cdot \mathbf{1}) + N(\mathbf{0}, \sigma^2 \mathbf{I}). \quad (31)$$

with  $\sigma^2 = 25$  and  $\gamma = 10$  – physically realistic values for these parameters. To generate Poisson noise, the `poissrnd` function in MATLAB’s Statistics

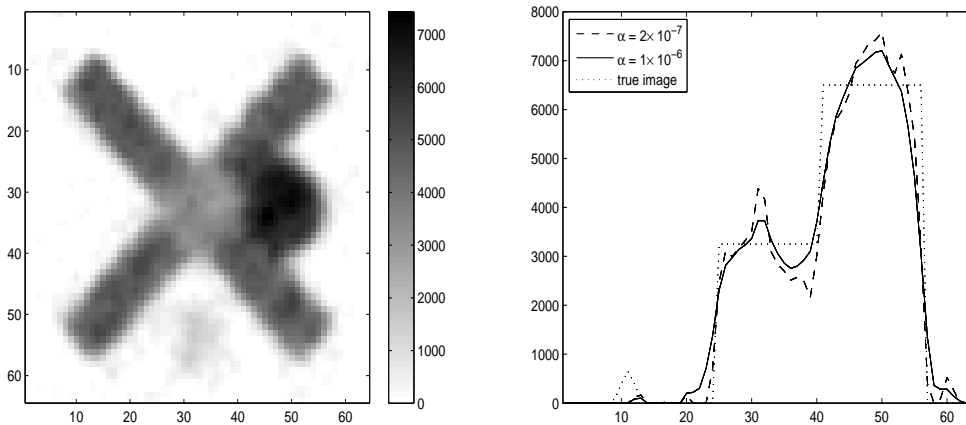


Figure 2. On the left is the reconstruction obtained by GPRNCG applied to (29) with  $\mathbf{L}$  the Laplacian and  $\alpha = 1 \times 10^{-6}$ . On the right is the 32nd row of the true image and the reconstructions with  $\alpha = 1 \times 10^{-6}$  and  $\alpha = 2 \times 10^{-7}$ , which minimizes the solution error.

Toolbox is used. The corresponding blurred, noisy data  $\mathbf{z}$  is given on the right hand side in Figure 1.

We use GPRNCG to obtain the solution of (29). The regularization parameter  $\alpha = 10^{-6}$  was chosen so that the solution error  $\|\mathbf{u}_\alpha - \mathbf{u}_{\text{exact}}\|$  was near to minimal ( $\alpha = 2 \times 10^{-7}$  minimized the solution error) but which yielded a reconstruction that was noticeable effected by the smoothing properties of the regularization function. This method of choosing  $\alpha$  is admittedly ad hoc. However, our objective in this paper is only to show that our method works in practice and that reconstructions are indeed smooth. The problem of automating regularization parameter choice is left for a later work. We stopped GPRNCG iterations after a 9 orders of magnitude decrease in the norm of projected gradient of  $T_\alpha$ . The reconstruction is given on the left in Figure 2. To demonstrate the effect of the Laplacian regularization, on the right in Figure 2, we plot the 32nd row of of the true image and the reconstructions with  $\alpha = 1 \times 10^{-6}$  and  $\alpha = 2 \times 10^{-7}$ , which minimizes the solution error.

#### 4 Conclusions

We have presented theoretical arguments justifying the use of regularization by diffusion for ill-posed Poisson likelihood estimation. In particular, we have shown that although minimizing  $T_0(u)$  over  $\mathcal{C}$  is an ill-posed problem,  $T_\alpha$  has a minimizer over  $\mathcal{C}$  for all  $\alpha > 0$  that depends continuously on the data  $z$  and

operator  $A$ . Moreover, we proved that there exists a positive sequence  $\{\alpha_n\}$  such that  $\alpha_n \rightarrow 0$  and  $u_{\alpha_n, n} \rightarrow u_{\text{exact}}$ , where  $u_{\alpha_n, n}$  is defined by the perturbed problem (10), provided  $A_n \rightarrow A$  and  $z_n \rightarrow z$ . We emphasize that these results are important in practice, because in applications one always deals with noisy data and a perturbed approximation of an (assumed) exact underlying model.

Following the theoretical arguments, we performed a numerical experiment that demonstrated the practical usefulness of the approach. In particular, we reconstructed a blurred image using a quadratic regularization function with a negative-Laplacian Hessian matrix. The results were clearly promising. We noted, furthermore, that our theory extends to the regularization approach set forth in [5, 11], which we hope to implement in the negative-log Poisson likelihood setting in a later work.

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