

Wavefront Estimation for Adaptive Optics Systems on Ground-Based Telescopes

Johnathan Bardsley*

Department of Mathematical Sciences, University of Montana, Missoula, MT, 59812, USA

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The classical approach for removing blur from an image d collected by a ground-based telescope is to solve a deconvolution problem of the form

$$d(x, y) = \int_{\mathbb{R}^2} k(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta \quad (1)$$

for f given data d and the *point spread function* (PSF) k . The idea behind adaptive optics (AO), however, is to “improve” the PSF k before the image is collected so that the resulting image will be of high quality.

In order to properly motivate the approach, we introduce the spatially invariant, Fourier optics PSF model (see Roggeman and Welch [5])

$$k[\phi](x, y) = \left| \mathcal{F}^{-1} \left\{ M(x, y) e^{i\phi(x, y)} \right\} \right|^2. \quad (2)$$

Here $M(x, y)$ is the telescope’s pupil indicator function, i.e. is 1 inside the pupil and 0 otherwise, and the function $\phi(x, y)$ denotes the phase error, or simply the phase, and is determined by the deviation from planarity of the incoming wavefronts of light at the point (x, y) . In Figure 1, we plot (2) in the *diffraction limited* case, i.e. when $\phi = 0$, and with a nonzero ϕ generated by a physically realistic model. Note the negative effect of nonzero phase error on the right-hand side PSF.

The job of the AO system is to remove the phase error from incoming wavefronts by introducing a counter wavefront ϕ_{DM} via, e.g., a deformable mirror (see Beckers [2]). Assuming the PSF has the form (2), the phase corrected PSF will have the form

$$k[\phi + \phi_{\text{DM}}](x, y) = \left| F^{-1} \left\{ M(x, y) e^{i(\phi + \phi_{\text{DM}})(x, y)} \right\} \right|^2. \quad (3)$$

Ideally, the deformable mirror created counter wavefront satisfies $\phi_{\text{DM}} = -\phi$, so that the resulting PSF has the *diffraction limited* form

$$k[0](x, y) = \left| F^{-1} \{ M(x, y) \} \right|^2, \quad (4)$$

in which case the *diffraction limited image*

$$d_{\text{DL}}(x, y) = \int_{\mathbb{R}^2} k[0](x - \xi, y - \eta) f(\xi, \eta) d\xi d\eta, \quad (5)$$

is what is seen by the telescope. This is the astronomers gold standard. In practice, however, an accurate approximation of $-\phi$ suffices for near diffraction limited imaging.

In ground-based astronomy, a phase error estimate is typically obtained from a measurement g of the wavefront gradient, which is assumed to satisfy the discrete model

$$g = \Gamma\phi + \eta.$$

Here ϕ denotes the unknown wavefront, Γ a discrete gradient operator and η a Gaussian random vector with zero mean and covariance matrix $\sigma^2 I$. Early approaches for estimating ϕ (see, e.g., [4]) involve computing a solution of the least squares normal equations

$$\Gamma^T \Gamma \phi = \Gamma^T g.$$

For large-scale adaptive optics systems, however, least squares solutions can be unstable, and hence, the minimum variance estimator is preferred [3]. Minimum variance estimation is a Bayesian statistical approach in which a prior probability density is assumed on the phase. In our case, it can be accurately assumed that ϕ is a realization of a Gaussian random vector with

* e-mail: bardsleyj@mso.umt.edu, Phone: +1 406 243 5328, Fax: +1 406 243 2674

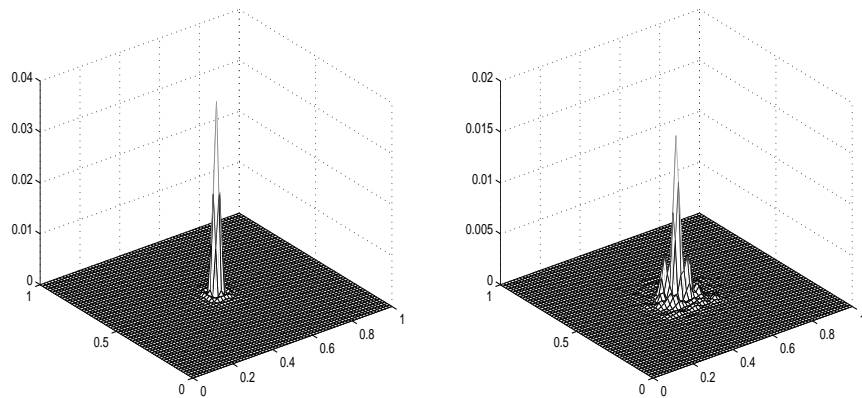


Fig. 1 Diffraction limited PSF (on the left), and PSF resulting from physically realistic phase error (on the right).

zero mean and known covariance matrix C_ϕ . The minimum variance estimator for ϕ is then the solution of the large sparse linear system

$$(\Gamma^T \Gamma + \sigma^2 C_\phi^{-1}) \phi = \Gamma^T g. \quad (6)$$

The problem of efficiently solving (6) has seen much recent attention. An efficient direct method using sparse matrix techniques is explored in [3]. However, the most computationally efficient approaches have involved the use of multigrid to precondition conjugate gradient iterations (MGPCG) [6]. Multigrid is an effective preconditioner in this case due to the fact that the coefficient matrix in (6) is approximately equal to the sum of discrete Laplacian ($\Gamma^T \Gamma$) and biharmonic (C_ϕ^{-1}) matrices. Multigrid is known to be very efficient in such cases.

In this talk, I introduced two new approaches. Both can be found, together with numerical demonstrations, in [1]. The first approach uses $\Gamma^T \Gamma + \sqrt{\epsilon} I$, where I is the identity matrix and ϵ is machine epsilon, as a preconditioner for conjugated gradient iterations applied to (6). I call this least squares preconditioned conjugate gradient (LSPCG). Since $\Gamma^T \Gamma + \sqrt{\epsilon} I$ is a sparse matrix a Cholesky factorization after a sparse reordering of indices can be computed. Then due to the fact that this matrix is fixed for a given adaptive optics system, the computation of the Cholesky factorization can be done off-line, making the application of the preconditioner very efficient.

In the second approach, it is noted that the solution of (6) is well approximated by the solution of

$$(I + \sigma^2 L) \phi = (\Gamma^T \Gamma + \sqrt{\epsilon} I)^{-1} \Gamma^T g, \quad (7)$$

where L is the discrete Laplacian matrix. This system can then be efficiently solved using the discrete Fourier transform. We note, moreover, that the minimum norm least squares solution on the right-hand side in equation (7) can be replaced with *any* least squares solution, with good results. I call this method gradient denoised least squares (GDLS).

A comparison of the two new methods with the MGPCG benchmark are given in [1]. I will summarize those results here. MGPCG yields solutions with the lowest relative error, but is the most expensive of the three methods to implement, based on CPU time comparisons in MATLAB. LSPCG is significantly more efficient than MGPCG, but does not yield reconstructions with as low of relative error. GDLS yields reconstructions with relative error nearly (but not quite) as low as those obtained by MGPCG and lower than those obtained by LSPCG. Furthermore, it is the most efficient of the three methods.

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