

A THEORETICAL FRAMEWORK FOR THE REGULARIZATION OF POISSON LIKELIHOOD ESTIMATION PROBLEMS

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ABSTRACT. Let $z = Au + \gamma$ be an ill-posed, linear operator equation. Such a model arises, for example, in both astronomical and medical imaging, in which case γ corresponds to background, u the unknown true image, A the forward operator, and z the data. Regularized solutions of this equation can be obtained by solving

$$R_\alpha(A, z) = \arg \min_{u \geq 0} \{T_0(Au; z) + \alpha J(u)\},$$

where $T_0(Au; z)$ is the negative-log of the Poisson likelihood functional, and $\alpha > 0$ and J are the regularization parameter and functional, respectively. Our goal in this paper is to determine general conditions which guarantee that R_α defines a *regularization scheme* for $z = Au + \gamma$. Determining the appropriate definition for *regularization scheme* in this context is important: not only will it serve to unify previous theoretical arguments in this direction, it will provide a framework for future theoretical analyses. To illustrate the latter, we end the paper with an application of the general framework to a case in which an analysis has not been done.

1. INTRODUCTION

Consider the linear operator equation

$$(1) \quad z = Au + \gamma.$$

Our application of interest is image processing, so that $z \in L^\infty(\Omega)$ denotes the image intensity, $u \in L^2(\Omega)$ the (nonnegative) intensity of the unknown object, and $\gamma \in L^\infty(\Omega)$ the (nonnegative) intensity of the background. Each function is defined on a closed, bounded domain $\Omega \subset \mathbb{R}^d$. Finally,

$$Au(x) \stackrel{\text{def}}{=} \int_{\Omega} a(x; y)u(y) dy,$$

where $a \in L^\infty(\Omega \times \Omega)$ is the error free, nonnegative point spread function (PSF). Given these assumptions, $A : L^2(\Omega) \rightarrow L^2(\Omega)$ is a compact operator, and hence, the problem of solving (1) for u is ill-posed [10, 11]. Moreover, $Au \geq 0$ whenever $u \geq 0$, and hence, assuming that the true image $u_{\text{exact}} \geq 0$, we have that the error free data $z = Au_{\text{exact}} + \gamma$ is bounded below by γ . Note that here, and in the remainder of the document, we omit “almost everywhere” from the mathematical statements in which its presence is called for.

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In previous works of the author [2, 3, 4], the following variational problem is theoretically analyzed:

$$(2) \quad \min_{u \in \mathcal{C}} \left\{ T_\alpha(Au; z) \stackrel{\text{def}}{=} T_0(Au; z) + \alpha J(u) \right\},$$

where

$$\mathcal{C} = \{u \in L^2(\Omega) \mid u \geq 0\},$$

α and J are the regularization parameter and functional, respectively, and

$$(3) \quad T_0(Au; z) \stackrel{\text{def}}{=} \int_{\Omega} ((Au + \gamma) - z \log(Au + \gamma)) dx.$$

T_0 is the functional analogue of the negative-log of the Poisson likelihood function [4, 7], which arises in positron emission tomography (PET) [8] and in image deblurring when a charge-coupled-device (CCD) camera is used to collect images [9].

In [2], an analysis of (2) with the regularization functional

$$(4) \quad J(u) = \|u\|_{L^2(\Omega)}^2,$$

where $\|u\|_{L^2(\Omega)}^2 = \int_{\Omega} u^2 dx$, is considered; whereas in [3], the regularization functional

$$(5) \quad J(u) = \|\mathbf{\Lambda}(x)\nabla u\|_{L^2(\Omega)}^2,$$

where $\mathbf{\Lambda}(x)$ is a 2×2 positive definite matrix with continuously differentiable components for all $x \in \Omega$, is the focus; and finally, in [4], we consider the total variation regularization functional

$$(6) \quad J(u) = J_\beta(u) \stackrel{\text{def}}{=} \sup_{v \in \mathcal{V}} \int_{\Omega} \left(-u \nabla \cdot v + \sqrt{\beta(1 - |v(x)|^2)} \right) dx,$$

where $\beta \geq 0$ and

$$\mathcal{V} = \{v \in C_0^1(\Omega; \mathbb{R}^d) : |v(x)| \leq 1 \quad x \in \Omega\}.$$

We note that if u is continuously differentiable on Ω , (6) takes the recognizable form [1, Theorem 2.1]

$$(7) \quad J_\beta(u) \stackrel{\text{def}}{=} \int_{\Omega} \sqrt{|\nabla u|^2 + \beta} dx.$$

$J_0(u)$ is known as the *total variation* of u .

In [2, 3, 4], the term *regularization scheme* is not defined. We will do this rigorously below, and then argue that the results of [2, 3, 4] imply that (2) defines a regularization scheme for J given by (4), (5), and (6). Moreover, the definition together with our previous arguments provide a general framework for future analysis of (2) with, for example, different regularization functionals, or, as we will see later, with convex transformations of T_0 , such as is used in positron emission tomography (PET) [8].

Before continuing, we note that the definition for T_0 used in [2, 3, 4] is given by $T_0(Au + \sigma^2; z + \gamma + \sigma^2)$, with T_0 defined by (3). We have made this change for two reasons: first, we are using (1) instead of $z = Au$ as our model, and second, we have removed the parameter σ^2 due to the fact that it stems from a noise model for CCD camera data [9] and, as such, is extraneous in the functional (non-stochastic and non-discrete) setting of this paper.

The paper is organized as follows. We present our definition of *regularization scheme* in Section 2. In Section 3, we prove that the results of [2, 3, 4] imply that problem (2) defines a regularization scheme when J is given by (4), (5), and (6).

Finally, in Section 4, we illustrate the utility of the general framework by applying it to a variational problem that arises in PET imaging.

2. DEFINITION OF REGULARIZATION SCHEME

We begin with some definitions and notation. Let

$$\mathcal{B} = \{\hat{a} \in L^\infty(\Omega \times \Omega) \mid \hat{a} \geq 0\} \times \{\hat{z} \in L^\infty(\Omega) \mid \hat{z} \geq 0\}.$$

\mathcal{B} is a closed subset of the Banach space $L^\infty(\Omega \times \Omega) \times L^\infty(\Omega)$, which has an induced norm given by $\|(\hat{a}, \hat{z})\|_{\mathcal{B}} \stackrel{\text{def}}{=} \max\{\|\hat{a}\|_{L^\infty(\Omega \times \Omega)}, \|\hat{z}\|_{L^\infty(\Omega)}\}$.

Next, we define the operator $R_\alpha : \mathcal{B} \rightarrow \mathcal{C}$ by

$$(8) \quad R_\alpha(a, z) \stackrel{\text{def}}{=} \arg \min_{u \in \mathcal{C}} T_\alpha(Au; z).$$

Finally, we define a sequence of operator equations

$$(9) \quad z_n = A_n u + \gamma,$$

where $z_n \in L^\infty(\Omega)$ is nonnegative, $u \in L^2(\Omega)$, $\gamma > 0$, and

$$A_n u(x) \stackrel{\text{def}}{=} \int_{\Omega} a_n(x; y) u(y) dy,$$

with $a_n \in L^\infty(\Omega \times \Omega)$ a nonnegative point spread function (PSF). Note, then, that $A_n u \geq 0$ whenever $u \in \mathcal{C}$.

The functions z_n and a_n should be viewed as approximations of the true data z and PSF a , respectively. In astronomy and PET, for example, *both* the data and PSF are estimated and hence contain errors. Thus our definition of regularization scheme, which we present now, accounts for errors in the measurements of both z and a .

Definition 2.1. $\{R_\alpha\}_{\alpha>0}$ defines a regularization scheme on \mathcal{B} provided

1. R_α is well-defined and continuous on \mathcal{B} for all $\alpha > 0$;
2. given a sequence $\{(a_n, z_n)\}_{n=1}^\infty \subset \mathcal{B}$ such that $\|(a_n, z_n) - (a, z)\|_{\mathcal{B}} \rightarrow 0$, there exists a positive sequence $\{\alpha_n\}_{n=1}^\infty$ such that

$$\|R_{\alpha_n}(a_n, z_n) - u_{\text{exact}}\|_{L^p(\Omega)} \rightarrow 0$$

for some $p \geq 1$.

In the next section, we show that $R_\alpha(a, z)$ satisfies conditions 1 and 2 of Definition 2.1 for the regularization functionals (4), (5), and (6).

3. $R_\alpha(a, z)$ DEFINES A REGULARIZATION SCHEME

3.1. $R_\alpha(a, z)$ IS A WELL-DEFINED OPERATOR ON \mathcal{B} . We begin by defining two function spaces that will be of import in the discussion that follows. First, recall the inner-product on the Hilbert space $H^1(\Omega)$ [5, 12] given by

$$(10) \quad \langle u, v \rangle_{H^1(\Omega)} \stackrel{\text{def}}{=} \langle u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)},$$

and $\|\cdot\|_{H^1(\Omega)} = \sqrt{\langle \cdot, \cdot \rangle_{H^1(\Omega)}}$. Next, the space of bounded variation [6] is defined

$$(11) \quad BV(\Omega) = \{u \in L^1(\Omega) : J_0(u) < +\infty\},$$

where J_0 is defined by (6). $BV(\Omega)$ is a Banach space with norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + J_0(u).$$

We now prove that solutions of (2) exist and are unique under certain reasonable assumptions. This will follow provided that T_α is strictly convex and coercive [5, 12].

In order to see that T_0 is convex, we note that the gradient and Hessian of T_0 are given, respectively, by

$$(12) \quad \nabla T_0(Au; z) = A^* \left(\frac{Au - (z - \gamma)}{Au + \gamma} \right),$$

$$(13) \quad \nabla^2 T_0(Au; z) = A^* \text{diag} \left(\frac{z}{(Au + \gamma)^2} \right) A,$$

where “ $*$ ” denotes operator adjoint. Given our assumptions regarding γ , z , and A , we see immediately that the Hessian is positive semi-definite for all $u \in \mathcal{C}$, and is positive definite provided the null-space of A is $\{0\}$. (Note that this significantly weakens the assumption of the invertibility of A made in [2, 4].) Moreover, we see that $\nabla T_0(Au_{\text{exact}}; z) = 0$, and hence, u_{exact} is a minimizer of T_0 .

The strict convexity of T_α requires a still weaker assumption on A if J is given by (4), (5), or (6). In particular, J given by (4) is strictly convex, in which case T_α is strictly convex for any A satisfying the assumptions made following (1). On the other hand, when J is given by (5) or (7), it is enough to assume that the null-space of A does not contain the constant functions (since then the null-spaces of $\nabla^2 T_0$ and $\nabla^2 J$ will not intersect). Thus we have that T_α is strictly convex if J is given by (4), or if J is given by (5) or (7) and the null-space of A does not contain the constant functions.

In addition to strict convexity, we need that T_α is coercive, which is defined

$$T_\alpha(Au; z) \rightarrow +\infty \quad \text{whenever} \quad \|u\| \rightarrow +\infty.$$

Here $\|\cdot\|$ is a norm that depends upon the choice of J : for J given by (4), $\|\cdot\| \stackrel{\text{def}}{=} \|\cdot\|_{L^2(\Omega)}$; for J given by (5), $\|\cdot\| \stackrel{\text{def}}{=} \|\cdot\|_{H^1(\Omega)}$; and for J given by (6), $\|\cdot\| \stackrel{\text{def}}{=} \|\cdot\|_{BV(\Omega)}$. Proofs of the coercivity of T_α for each case can be found in [2, 3, 4].

It follows, then, that solutions of (2) exist and are unique—making $R_\alpha(a, z)$ a well-defined operator—when J is given by (4), or when J is given by (5) or (7) and the null-space of A does not contain the constant functions.

Another question of interest, which has not been addressed directly in our previous work, is: what is the range of R_α ? The proofs of existence and uniqueness of minimizers of T_α found in [2, 3, 4] indicate that when J is given by (4), $\text{Range}(R_\alpha) \subset L^2(\Omega)$; when J is given by (5), $\text{Range}(R_\alpha) \subset H^1(\Omega)$; and when J is given by (6), $\text{Range}(R_\alpha) \subset BV(\Omega)$. Thus the choice of regularization functional has a significant effect on the properties of the regularized solution, as is readily verified by numerical experiment (see [2, 3, 4]).

3.2. $R_\alpha(a, z)$ IS A CONTINUOUS OPERATOR ON \mathcal{B} . For each choice of regularization functional J , the theoretical arguments in [2, 3, 4] give us that $\|R_\alpha(a_n, z_n) - R_\alpha(a, z)\|_{L^p(\Omega)} \rightarrow 0$, for some $p \geq 1$, provided $\|A_n - A\|_{L^1(\Omega)}, \|z - z_n\|_{L^\infty(\Omega)} \rightarrow 0$. Noting that

$$\begin{aligned} \|(\hat{A} - A)u\|_{L^1(\Omega)} &= \int_{\Omega} \left| \int_{\Omega} (\hat{a}(x, y) - a(x, y))u(y) dy \right| dx \\ &\leq \|\hat{a} - a\|_{L^\infty(\Omega \times \Omega)} \cdot \|u\|_{L^1(\Omega)}, \end{aligned}$$

we see that

$$\|\hat{A} - A\|_{L^1(\Omega)} = \max_{u \in L^1(\Omega)} \frac{\|(\hat{A} - A)u\|_{L^1(\Omega)}}{\|u\|_{L^1(\Omega)}} \leq \|\hat{a} - a\|_{L^\infty(\Omega \times \Omega)},$$

and hence, $\|\hat{a} - a\|_{L^\infty(\Omega \times \Omega)} \rightarrow 0$ implies $\|\hat{A} - A\|_{L^1(\Omega)} \rightarrow 0$. Thus we have that if $\|(a_n, z_n) - (a, z)\|_{\mathcal{B}} \rightarrow 0$ then $\|R_\alpha(a_n, z_n) - R_\alpha(a, z)\|_{L^p(\Omega)} \rightarrow 0$, where $p = 2$ when J is given by (4) or (5), and $1 \leq p < d/(d-1)$ when J is given by (6), where d is the dimension of the computational domain.

However, we have found an error in the continuity arguments found in [2]. In particular, these arguments only give the result that $R_\alpha(a_n, z_n)$ converges to $R_\alpha(a, z)$ weakly in $L^2(\Omega)$, but strong convergence in $L^2(\Omega)$ follows if $\|R_\alpha(a_n, z_n)\|_{L^2(\Omega)} \rightarrow \|R_\alpha(a, z)\|_{L^2(\Omega)}$. We prove this now. By the weak lower semi-continuity of the $L^2(\Omega)$ norm $\|R_\alpha(a, z)\|_{L^2(\Omega)} \leq \liminf \|R_\alpha(a_n, z_n)\|_{L^2(\Omega)}$. However

$$T_\alpha(A_n(R_\alpha(a_n, z_n)), z_n) \leq T_\alpha(A_n(R_\alpha(a, z)), z_n)$$

implies

$$\begin{aligned} \liminf \|R_\alpha(a_n, z_n)\|_{L^2(\Omega)}^2 &\leq \|R_\alpha(a, z)\|_{L^2(\Omega)}^2 + 2\alpha^{-1}[T_0(A_n(R_\alpha(a, z))) \\ &\quad - \liminf T_0(A_n(R_\alpha(a_n, z_n)), z_n)] \\ &\leq \|R_\alpha(a, z)\|_{L^2(\Omega)}^2, \end{aligned}$$

since T_0 is weakly, lower semi-continuous. Hence $\|R_\alpha(a_n, z_n)\|_{L^2(\Omega)}$ converges strongly to $\|R_\alpha(a, z)\|_{L^2(\Omega)}$, which gives us the result.

Thus R_α satisfies condition 1 of Definition 2.1 for J given by (4), (5), and (6).

3.3. $R_\alpha(a, z)$ IS CONVERGENT. Finally, we must show that condition 2 of Definition 2.1 holds. First, we note that by our discussion above, if $\|(a_n, z_n) - (a, z)\|_{\mathcal{B}} \rightarrow 0$ then $\|A_n - A\|_{L^1(\Omega)}, \|z_n - z\|_{L^\infty(\Omega)} \rightarrow 0$.

In [2, 3, 4], it is argued that if $\|A_n - A\|_{L^1(\Omega)}, \|z - z_n\|_{L^\infty(\Omega)} \rightarrow 0$ and $\{\alpha_n\}_{n=1}^\infty$ is chosen so that $\alpha_n \rightarrow 0$ at a rate such that

$$(14) \quad (T_0(A_n u_{\text{exact}}; z_n) - T_0(A_n u_{0,n}; z_n)) / \alpha_n \text{ is bounded,}$$

where $u_{0,n}$ is a minimizer of $T_0(A_n u; z_n)$ over \mathcal{C} , it follows that

$$(15) \quad \|R_{\alpha_n}(A_n, z_n) - u_{\text{exact}}\|_{L^p(\Omega)} \rightarrow 0,$$

for $p = 2$ when J is given by (4) or (5), and for $1 \leq p < d/(d-1)$ when J is given by (6). Then R_α satisfies condition 2 of Definition 2.1. However, in [2, 3, 4], the existence of $u_{0,n}$ is not argued. Rather than do this, we can replace (14) by

$$(16) \quad \left(T_0(A_n u_{\text{exact}}; z_n) - \inf_{u \in \mathcal{C}} T_0(A_n u; z_n) \right) / \alpha_n \text{ is bounded,}$$

and the arguments of the proofs in [2, 3, 4] go through unchanged. We note that $\inf_{u \in \mathcal{C}} T_0(A_n u; z_n)$ exists due to the fact that $T_0(A_n u; z_n)$ is bounded below. This follows from Jensen's inequality (assuming w.l.o.g. $|\Omega| = 1$) and the properties of the function $x - c \log x$ for $c > 0$:

$$\begin{aligned} T_0(A_n u; z_n) &\geq \|A_n u + \gamma\|_1 - \|z_n\|_\infty \log \|A_n u + \gamma\|_1, \\ &\geq \|z_n\|_\infty - \|z_n\|_\infty \log \|z_n\|_\infty. \end{aligned}$$

For J given by (4), however, the arguments of [2] remain incomplete. In particular, from those arguments, we have only that $R_{\alpha_n}(a_n, z_n)$ converges to u_{exact}

weakly in $L^2(\Omega)$. But strong convergence can be proved provided (16) is replaced by

$$(17) \quad \left(T_0(A_n u_{\text{exact}}; z_n) - \inf_{u \in \mathcal{C}} T_0(A_n u; z_n) \right) / \alpha_n \rightarrow 0.$$

This change does not effect the other arguments since (17) implies (16). Now, to prove strong convergence, we note, as above, that weak convergence in $L^2(\Omega)$, together with $\|R_{\alpha_n}(a_n, z_n)\|_{L^2(\Omega)} \rightarrow \|u_{\text{exact}}\|_{L^2(\Omega)}$, implies strong convergence in $L^2(\Omega)$. To prove norm convergence, we note that

$$(18) \quad \inf_{u \in \mathcal{C}} T_0(A_n u, z_n) \leq T_{\alpha_n}(A_n(R_{\alpha_n}(a_n, z_n)), z_n),$$

and that by the weak lower semi-continuity of the norm,

$$\|u_{\text{exact}}\|_{L^2(\Omega)}^2 \leq \liminf \|R_{\alpha_n}(a_n, z_n)\|_{L^2(\Omega)}^2.$$

Thus by (18) together with (17), we have

$$\begin{aligned} \liminf \|R_{\alpha_n}(a_n, z_n)\|_{L^2(\Omega)}^2 &\leq \|u_{\text{exact}}\|_{L^2(\Omega)}^2 + 2 \liminf (\alpha_n^{-1} [T_0(A_n u_{\text{exact}}, z_n) \\ &\quad - T_0(A_n(R_{\alpha_n}(a_n, z_n)), z_n)]) \\ &\leq \|u_{\text{exact}}\|_{L^2(\Omega)}^2 + 2 \liminf (\alpha_n^{-1} [T_0(A_n u_{\text{exact}}, z_n) \\ &\quad - \inf_{u \in \mathcal{C}} T_0(A_n u, z_n)]) \\ &= \|u_{\text{exact}}\|_{L^2(\Omega)}^2. \end{aligned}$$

Hence we have $\|R_{\alpha_n}(a_n, z_n)\|_{L^2(\Omega)}^2 \rightarrow \|u_{\text{exact}}\|_{L^2(\Omega)}^2$ and so $R_{\alpha_n}(a_n, z_n)$ converges to u_{exact} strongly in $L^2(\Omega)$.

Thus R_α satisfies condition 2 of Definition 2.1 for J given by (4), (5), and (6).

3.4. CONCLUSIONS. The discussion above constitutes a proof of the following theorem.

Theorem 3.1. *Let $R_\alpha : \mathcal{B} \rightarrow \mathcal{C}$ be defined as in (8). Then $\{R_\alpha\}_{\alpha>0}$ is a regularization scheme, as defined in Definition 2.1, when J is given by (4), and when J is given by (5) or (6) and the null-space of A does not contain the constant functions. Moreover, if J is given by (4) then $\text{Range}(R_\alpha) \subset L^2(\Omega)$; if J is given by (5) then $\text{Range}(R_\alpha) \subset H^1(\Omega)$; and if J is given by (6) then $\text{Range}(R_\alpha) \subset BV(\Omega)$.*

We advocate the use of Definition 2.1 for use in proving analogous results for (8) with regularization functionals other than those presented in this paper, or with modifications of the likelihood function. We present such an example next.

4. USING THE FRAMEWORK ON A NEW EXAMPLE

In positron emission tomography (PET), the following model is sometimes used in place of (1) [8]:

$$z = z_0 e^{-Au} + \gamma,$$

where $z_0 \in L^\infty(\Omega)$ is known as the blank scan and is positive, and z , A , u , and γ are as above. The corresponding variational problem then takes the form

$$(19) \quad R_\alpha(a, z) \stackrel{\text{def}}{=} \arg \min_{u \in \mathcal{C}} \{T_0(z_0 e^{-Au}, z) + \alpha J(u)\},$$

where T_0 is defined as in (3), and the regularization function J is one of (4), (5), or (6). Since (19) results from a strictly convex (and hence continuous) transformation

of T_0 , condition 1 of Definition 2.1 immediately follows from the same result for (8). Condition 2 of Definition 2.1 follows for (19) from the fact that $\|e^{-Au} - e^{-Av}\|_1 \leq \|Au - Av\|_1$ for all $u, v \in \mathcal{C}$ (this can be easily verified by an appeal to the inequality that is needed in the proofs of convergence in each of [2, 3, 4]). Thus Theorem 3.1 also holds for (19).

5. CONCLUSIONS

We have provided a definition of regularization scheme for (1) and have shown that the arguments of [2, 3, 4] (together with some corrections made here) imply that (8) defines a regularization scheme for J given by (4), (5), and (6). This result is given in detail in Theorem 3.1, and we note that we have significantly weakened the assumptions made on the operator A in [2, 3, 4]. We also demonstrate the usefulness of the general framework implied by the definition and our previous arguments by using them to prove, in a straightforward manner, that the results of Theorem 3.1 also hold for (19) which arises in positron emission tomography.

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